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ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 245–250
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

The recovery of the Brunt–Väisälä frequency using dispersion curves☆

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Received 17 September 2004

Abstract

The coefficients of the Taylor expansion of the square of the Brunt–Väisälä frequency (BVF) are found, assuming its analyticity and certain symmetry conditions, using a well-known model [Miropol'skii YuZ. *The Dynamics of Internal Gravitational Waves in are Ocean*. Leningrad: Gidrometeoizdat; 1981.] employing an integral equation for the BVF constructed using a sequence of dispersion curves of internal gravitational waves in a vertically stratified ocean of constant depth. Assuming that the square of the BVF can be represented in the form of a fourth-order polynomial of the depth of the liquid layer being considered, it is shown that no more than two such polynomials correspond to one and the same sequence of dispersion curves. The coefficients of these polynomials are found.

The problem of recovering the Brunt–Väisälä frequency (BVF) using dispersion curves has become urgent after advances in the interpretation of the pattern of internal waves which propagate on the surface of the ocean in the form of luminous bright spots which move at a definite velocity. It is possible to find the equations of the first few dispersion curves approximately from observations. There is no one-to-one correspondence between the set of sequences of dispersion curves and the set of BVFs, and the problem of finding the set of BVFs which correspond to a specified sequence of dispersion curves in a sufficiently wide class of BVFs is therefore of interest. Some problems touching on this theme were considered earlier in Refs. 2–7.

1. Formulation of the problem

The problem of recovering the Brunt–Väisälä frequency (BVF) for a vertically stratified ocean of constant depth using a sequence of dispersion curves for internal gravitational waves is considered. The model in Ref. 1 is taken as the mathematical model for such an ocean. This model reduces to the following boundary-value problem

$$w'' - \frac{\mu(z)}{g}w' + \frac{\mu(z)-\omega^2}{\omega^2-f^2}k^2 w = 0, \quad w = w(z), \quad z \in [-H, 0]$$
$$w(-H) = 0, \quad w'(0) = g\frac{k^2}{\omega^2-f^2}w(0) \tag{1.1}$$

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 2, pp. 276–281, 2006.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2006.06.006

where w is the amplitude function of the vertical component of the velocity of the liquid particles, $\mu(z)$ is the square of the BVF, H is the depth of the ocean, g is the acceleration due to gravity, f is the Coriolis parameter, k is the wave number and ω is the frequency of free harmonic waves.

Specific problems of recovering the BVF using dispersion curves in certain classes have been considered previously. In particular, an algorithm was proposed in Ref. 6 for re-establishing the VBF using a sequence of dispersion curves in the case when it is known in an interval $[-H, -aH]$, $a \in (0, 1)$, and examples of parametric classes are presented in which the BVF is recovered uniquely.

In this paper we consider the problem of recovering the analytic function $\mu(z)$, which is symmetrical about the middle of the interval $[-H, 0]$, and also the case when $\mu(z)$ is a fourth-order polynomial. In the first case, the VBF is determined uniquely from a sequence of dispersion curves, and in the second case it is shown that there cannot be more than two such polynomials. As an example of the theory constructed, we also consider the recovery of the BVF in the class of second-degree polynomials. It is shown that there can also be no more than two such polynomials.

2. Basic relations

Changing to dimensionless quantities, new parameters and a new function

$$x = -\frac{z}{H}, \quad u = \frac{w}{(gH)^{1/2}}, \quad \eta^2 = \frac{H}{g}\omega^2, \quad \xi^2 = H^2k^2, \quad q(x) = \frac{g}{H}\mu(-Hx)$$

$$F^2 = \frac{H}{g}f^2, \quad \lambda = \frac{\xi^2}{\eta^2 - F^2}, \quad s = \lambda^2 - \lambda F^2 - \xi^2, \quad \upsilon = u\exp(\lambda x)$$

we change from the boundary-value problem (1.1) to the problem

$$\upsilon'' + (q(x) - 2\lambda)\upsilon' = -s\upsilon; \quad \upsilon'(0) = 0, \quad \upsilon(1) = 0 \tag{2.1}$$

The boundary-value problem (2.1) is equivalent to the integral equation

$$z(x) \triangleq \upsilon(x)\sqrt{\rho(x)}\exp(-\lambda x) = s\int_0^1 K(x, t, \lambda)z(t)dt \tag{2.2}$$

with symmetric kernel

$$K(x, t, \lambda) = \exp[-\lambda(x+t)]\sqrt{\rho(x)\rho(t)}\int_\sigma^1 \frac{\exp(2\lambda y)}{\rho(y)}dy$$

$$\sigma = \begin{cases} x, & t < x \\ t, & t > x \end{cases}, \quad \rho(x) = \exp\left(\int_0^x q(t)dt\right)$$

For any real λ, the kernel $K(x, t, \lambda)$ satisfies the oscillatory conditions in Finkel'shtein's theorem and, consequently, is oscillatory. The integral Eq. (2.2) therefore has a denumerable set of positive and simple characteristic values $s_0(\lambda) < s_1(\lambda) < \ldots$. The functions $s_j(\lambda)$ $(j = 0, 1, \ldots)$ can be determined from the dispersion curves of problem (1.1).[6]

We next obtain the integral equation for the dimensionless BVF, that is, for the function $q(x)$. To do this, we consider the trace of the kernel $K(x, t, \lambda)$

$$A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{s_n(\lambda)} \tag{2.3}$$

It can be represented as

$$A(\lambda) = \int_0^1 K(x, x, \lambda)dx = \int_0^1 \exp(2\lambda x)\Phi(x)dx; \quad \Phi(x) = \int_x^1 \frac{\rho(t-x)}{\rho(t)}dt \tag{2.4}$$

The function $\Phi(x)$ is expressed in terms of $A(\lambda)$ using an inverse Laplace transform and, in this way, is defined by the sequence of dispersion curves since, according to formula (2.3), the trace $A(\lambda)$ is determined by it. Writing the function $\Phi(x)$ in the form

$$\Phi(x) = \int_x^1 \exp\left(\int_t^{t-x} q(y)dy\right)dt \tag{2.5}$$

we obtain the basic integral equation for $q(x)$.

It can be verified that, if the function $q(x)$ is a solution of Eq. (2.5), then the function $q(1-x)$ is also a solution of this equation. Furthermore, we note that, if $\upsilon_j(x)$ is an eigenfunction of problem (2.1) which corresponds to an eigenvalue $s_j(\lambda)$, then the function

$$w_j(x) = \int_0^{1-x} \rho(t)\upsilon_j(t)\exp(-2\lambda t)dt$$

will be an eigenfunction of the boundary-value problem

$$\upsilon'' + (q(1-x) - 2\lambda)\upsilon' = -s\upsilon; \quad \upsilon'(0) = 0, \quad \upsilon(1) = 0 \tag{2.6}$$

which corresponds to the very same eigenvalue $s_j(\lambda)$. It follows from this that the boundary-value problems (2.1) and (2.6) have one and the same eigenvalues. Consequently, the inverse eigenvalue problem for boundary value problem (2.1) has at least two solutions $q_1(x)$ and $q_2(x)$ which are related by the condition $q_1(x) = q_2(1-x)$.

3. Recovery of the function $q(x)$ using a sequence of dispersion curves in the case when the function $q(x)$ is symmetrical about the middle of the interval [0,1] and analytic in the circle $|z| < 1$

We will assume that the function $q(x) \in A_1$ and that it satisfies the condition $q(x) = q(1-x)$. We now find the coefficients of the Taylor's expansion of the function $q(x)$ which satisfies Eq. (2.5) in this case. We consider the function $F(x) = \Phi(1-x)$. After a change of variable under the integral sign in the expression for the function $F(x)$, we obtain the representation

$$F(x) = \int_0^x \exp\left(\int_0^{x-u} q(y)dy\right)\exp\left(\int_1^u q(1-y)dy\right)du \tag{3.1}$$

It has been shown in Ref. 6 that

$$\int_0^1 q(t)dt = -1 - \Phi'(0)$$

On introducing the function $f(x) = F(x)\exp(-1 - \Phi'(0))$ and taking account of the fact that $q(y) = q(1-y)$, we obtain from relation (3.1) that

$$f(x) = \int_0^x \rho(u)\rho(x-u)du \tag{3.2}$$

We now expand the functions $\rho(u)$ and $f(x)$ in the power series

$$\rho(u) = \sum_{k=0}^{\infty} c_k u^k, \quad f(x) = \sum_{n=0}^{\infty} f_n x^n$$

Note that $c_0 = 1, f_0 = 0, f_1 = 1$. Substituting the power-series expansions into expression (3.2) and equating coefficients of like powers of x, we obtain the following system of equations for finding the coefficients $c_k (k = 0, 1, \ldots)$

$$f_n = \sum_{k+j+1=n} c_k c_j \frac{k!j!}{n!}, \quad n = 1, 2, \ldots$$

The first equation gives the equality $c_0^2 = f_1$. A value of unity has to be taken for c_0. From the second equation, we obtain $c_1 = f_2$. The subsequent coefficients are found recurrently from the equalities

$$c_{n-1} = \frac{n}{2}\left[f_n - \sum_{k=1}^{n-2} c_k c_{n-k-1} \frac{k!(n-k-1)!}{n!}\right], \quad n = 3, 4, \ldots \tag{3.3}$$

Hence, in the case under consideration, the function $q(x) = \rho'(x)[\rho(x)]^{-1}$ is uniquely recovered from a sequence of dispersion curves using formulae (3.3).

4. Finding solutions of the main equation in the class of fourth-order polynomials

Noting that, if $q(x)$ is a polynomial of degree n then $\Phi(x)$ is an $(n+1)$-th order integral function, it is possible to find the degree of the polynomial $q(x)$ by establishing the order of growth of the function $\Phi(x)$.

We will consider the case when the required polynomial is of the fourth order. Since the function $q(1-x)$ is also a fourth-order polynomial, the main equation is solvable, generally speaking, in more than one way. It will next be shown that there are no more than two solutions.

Suppose

$$q(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon, \quad \alpha \neq 0$$

The problem involves finding the coefficients of this polynomial using the function $\Phi(x)$. Substituting the expression for $q(x)$ into relation (2.5), after some obvious reduction we have

$$\Phi(x) = x\int_1^{\frac{1}{x}} \exp\left\{\frac{\alpha}{5}x^5[(u-1)^5 - u^5 + k(x, u)]\right\}du$$

where the function $k(x, u)$ uniformly tends to zero for $u \in [-1, 1]$ as $x \to \infty$.

The coefficient α can be found using the asymptotic behaviour of the function $\Phi(x)$ for large values of the argument. To do this, it is first necessary to determine the sign of α, which is established from whether the function $\Phi(x)$ tends to zero or infinity as $x \to +\infty$. This follows from the fact that

$$\lim_{x\to+\infty} \Phi(x) = 0 \quad \text{when} \quad \alpha > 0, \qquad \lim_{x\to+\infty} \Phi(x) = -\infty \quad \text{when} \quad \alpha < 0$$

Having established the sign of α, we find the value of α using Laplace's asymptotic method,

$$\alpha = -80\lim(x^{-5}\ln|\Phi(x)|) \quad \text{when} \quad x\,\mathrm{sign}\,\alpha \to +\infty$$

To find the remaining coefficients, we use corollaries obtained from the main equation by direct differentiation[6,a]

$$q(0) + q(1) = \Phi_1, \quad \int_0^1 q^n(t)dt = \Phi_{n+1}, \quad n = 1, 2, \quad q'(0) - q'(1) - q(0)q(1) = \Phi_4$$
$$q''(0) + q''(1) - 3q(1)q'(1) + 3q(0)q'(0) - q(0)q'(1) + q(1)q'(0) - 2\Phi_1 q(0)q(1) = \Phi_5 \tag{4.1}$$

[a] See also: Ryndina, V. V. Coefficients of the stratification parameter of a liquid in the class of polynomials of no higher than the fourth order. Rostov-on-Don, 2002. Deposited in the All-Union Institute for Scientific and Technical information (VINITI) 18.10.02. No. 1778-B02.

where

$$\Phi_1 = P\Phi''(1),\quad \Phi_2 = -1-\Phi'(0),\quad \Phi_3 = \Phi''(0)-\Phi_1$$

$$\Phi_4 = -\Phi_1^2 - P\Phi^{(3)}(1),\quad \Phi_5 = P\Phi^{(4)}(1)-\Phi_1^3;\quad P = -[\Phi'(1)]^{-1}$$

Writing equalities (4.1) in terms of the coefficients of the polynomial $q(x)$, we obtain five equations which these coefficients must satisfy.

From the first, second and fourth equalities of (4.1), we obtain the three equations

$$\alpha+\beta+\gamma+\delta+2\varepsilon = \Phi_1,\quad \frac{1}{5}\alpha+\frac{1}{4}\beta+\frac{1}{3}\gamma+\frac{1}{2}\delta+\varepsilon = \Phi_2$$

$$4\alpha+3\beta+2\gamma+\varepsilon(\Phi_1-\varepsilon) = -\Phi_4$$

Putting

$$T = \varepsilon(\Phi_1-\varepsilon) \tag{4.2}$$

we obtain from these equations that

$$\alpha = \alpha_0-\frac{5}{2}T,\quad \beta = \beta_0+4T+2(\delta+2\varepsilon),\quad \gamma = \gamma_0-\frac{3}{2}T-3(\delta+2\varepsilon) \tag{4.3}$$

The constants

$$\alpha_0 = -15\Phi_1+30\Phi_2-\frac{5}{2}\Phi_4,\quad \beta_0 = 28\Phi_1-60\Phi_2+4\Phi_4,\quad \gamma_0 = -12\Phi_1+30\Phi_2-\frac{3}{2}\Phi_4$$

are uniquely defined by the function $\Phi(x)$ and, consequently, also by the sequence of dispersion curves.

From the first equation of (4.3), we obtain that $T=2/5(\alpha_0-\alpha)$. Since α, as was shown above, is uniquely determined by the sequence of dispersion curves, then T is also uniquely determined by the sequence of dispersion curves. Generally speaking, Eq. (4.2) in ε determines two values: ε_1 and ε_2. Substituting expressions (4.3), instead of the coefficients α, β and γ of the polynomial $q(x)$, and using the first and fifth equalities of (4.1), we obtain the equality

$$2\delta(\Phi_1-2\varepsilon) = \Phi_6-2\varepsilon(\Phi_4+T) \tag{4.4}$$

where

$$\Phi_6 = -60\Phi_1+120\Phi_2-12\Phi_4+3\Phi_1\Phi_4+T(\Phi_1-12)-\Phi_5$$

If the roots of Eq. (4.2) are different, then $\Phi_1-2\varepsilon_j\neq 0$ ($j=1, 2$), and, from Eq. (4.4), we obtain two values of δ which correspond to the two different values of ε:

$$\delta_j = \frac{\Phi_6-2\varepsilon_j(\Phi_4+T)}{2(\Phi_1-2\varepsilon_j)},\quad j = 1, 2$$

In this case, we obtain the corresponding values of β_j, γ_j ($j=1, 2$) from formulae (4.3). Hence, if the roots of Eq. (4.2) are different, then two functions

$$q_j(x) = \alpha x^4+\beta_j x^3+\gamma_j x^2+\delta_j x+\varepsilon_j,\quad j = 1, 2$$

exist which satisfy Eq. (2.5).

If $\varepsilon_1=\varepsilon_2$, then $\varepsilon_j=\Phi_1/2$. Hence $\Phi_1-2\varepsilon_j=0$, and the quantity δ is not determined from the equality (4.4). In this case, to find the coefficient δ we invoke the third equality of (4.1) and obtain

$$\int_0^1[P_1(x)+\delta(2x^3-3x^2+x)]^2dx = \Phi_3$$

$$P_1(x) = \alpha x^4+\beta_1 x^3+\gamma_1 x^2+\frac{1}{2}\Phi_1;\quad \beta_1 = \beta_0+4T+2\Phi_1,\quad \gamma_1 = \gamma_0-\frac{3}{2}T-3\Phi_1 \tag{4.5}$$

The polynomial $P_1(x)$ is uniquely defined by the sequence of dispersion curves.

From relation (4.5), we obtain the following quadratic equation for δ

$$\frac{1}{210}\delta^2 + 2\theta_1\delta + \theta_2 = \Phi_3$$
$$\theta_1 = \int_0^1 (2x^3 - 3x^2 + x)P_1(x)dx, \quad \theta_2 = \int_0^1 P_1^2(x)dx \tag{4.6}$$

from which, generally speaking, the two roots δ_1 and δ_2 are determined. In this case, we obtain the corresponding values of β_j, $\gamma_j (j = 1, 2)$ from formulae (4.3), that is, in the case when $\varepsilon_1 = \varepsilon_2$, two solutions of Eq. (2.5) exist if $\delta_1 \neq \delta_2$ and one solution if $\delta_1 = \delta_2$.

With the same sequence of dispersion curves, no other fourth-order polynomials exist, apart from those found above.

When $q(x)$ turns out to be a polynomial of lower than the fourth order, its coefficients can be found from the formulae obtained above by equating the corresponding coefficients to zero. However, it is simpler to carry out an independent solution of the appropriate system of equations which follows from the main equation (2.5). Thus, in the class of second-order polynomials

$$q(x) = \alpha x^2 + \beta x + \gamma; \quad \alpha \neq 0, \quad \alpha, \beta, \gamma \in R$$

from the first, second and fourth equalities of (4.1) we obtain a system of equations for finding the coefficients α, β and γ, the solution of which leads, generally speaking, to two functions constituting the solution of Eq. (2.5):

$$q_j(x) = \alpha x^2 - (\alpha - (-1)^j\sqrt{L})x + \frac{1}{2}(\Phi_1 - (-1)^j\sqrt{L}), \quad j = 1, 2$$
$$\alpha = 3\Phi_1 - 6\Phi_2, \quad L = \Phi_1^2 + 8\alpha + 4\Phi_4$$

A detailed investigation of the solutions of Eq. (2.5) in the class of third-order polynomials was carried out earlier.[b]

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Translated by E.L.S.

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